tion of wall lining. For the ducts shown, the impedance of the duct walls could assume ten or twelve different values and a standard minimization routine would compute the combination of ten or twelve impedances yielding the greatest attenuation of sound levels within the duct. In actual practice, to limit computation time, one requires the impedance to be constant on an entrance section of the duct (made up of, say, four integration points), then permits the impedance to change to a second value on a middle section (comprised of four or more integration points), and then have a third value on a termination section (consisting of another four integration points). Then the minimization routine minimizes a function of three complex or six real variables, a much less time-consuming effort than minimizing a function of 24 real variables.

To compare the computation times of the integral equation approach versus that of the finite-difference approach, a uniform duct with $L/H=0.5$ (L and H are the duct length and height), $\eta=1$, and $\zeta_w=0.388-0.846i$ was analyzed. The study included different sets of integration points around the boundary of the duct for the integral equation method and 100 and 400 mesh points for the finite-difference method. The results are summarized in Table 1.

The conclusion drawn from an inspection of Table 1 is that the integral equation method can give comparable accuracy with much less computational time than the finite-difference method. A key factor is that the finite-difference method computes the pressure distribution within the entire duct even though the acoustic intensities are required only at the entrance and the exit. However, in the integral equation method, an appreciable savings in computational time is obtained by computing only the desired quantities at the entrance and the exit.

For the optimizations to be carried out, the setup time occurred only once even if 50 different impedance values were considered. Hence, the time required is less than 1 sec per evaluation or 50 sec total for an optimization requiring 50 function evaluations. This compares with 300 sec of computation time for a comparable study using finite differences. All the computations were carried out on a CDC 6600 computer.

The results for three of the ducts depicted in Fig. 1 are summarized in Table 2. The conical duct was chosen because the exact solution is known for a hard walled duct. The tapered duct was chosen because it has not been handled by the conformal mapping/finite-difference method and is extremely easy to handle using integral equations. The catenoidal duct is more difficult. It has not been solved by the conformal mapping/finite-difference method either. The equation of the outside wall is straightforward because it is a catenary. However, the exit curve (or surface) is more difficult and is dealt with numerically.

For all three example cases considered, a three-sectional liner was found which yielded twice to three times the attenuation of the best uniform liner. This agrees with results obtained in Ref.2.

Appendix: Fundamental Solutions of the Partial Differential Equations Arising in Duct Acoustics

The fundamental solution for the partial differential equation of duct acoustics for a duct containing mean flow, that is, for an equation of the form

$$(1-M^2)\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}-4\pi\eta Mi\frac{\partial p}{\partial x}+(2\pi\eta)^2 p=0 \qquad \text{(A1)}$$

is

$$p(x,y)=e^{(ax)}H_0^1(wr) \qquad \text{(A2)}$$

where

$$a=i2\pi\eta M/(1-M^2)$$

$$w=(2\pi\eta)^2+a^2(1-M^2)-i4\pi\eta Ma$$

$$r=[(x/[1-m^2]^{1/2}-s)^2+(y-t)^2]^{1/2}$$

$$H_0^1=J_0+iY_0$$

(See Eq. (4) and Ref. 12). This is verified by direct substitution of Eq. (A2) into Eq. (A1).

Acknowledgment

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Derivation of an Integral Equation for Transonic Flows

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I. Introduction

THE nonlinear partial differential equation for the perturbation velocity potential and the boundary conditions

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which describe steady, inviscid, compressible, transonic flow past a thin two-dimensional airfoil can be transformed into a singular integro-differential equation. Differentiation of the integro-differential equation yields an integral equation. At the present time, two forms of this integral equation exist in the literature, one for the singularity that is enclosed in an infinitely long strip of vanishing thickness[1] and the other for the singularity that is enclosed in a vanishing circle.[2] In this article, we derive a more general integral equation by enclosing the singularity in a vanishing rectangular cavity of arbitrary aspect ratio, and we deduce the two integral equations commonly used as special cases.

II. Mathematical Preliminary

In Sec. II, we present the essential mathematical concepts:

a) If f is a function which becomes infinite at a point P within the surface S bounded by a sectionally smooth curve C, surround P by a cavity σ, and define the integral of f through the whole surface S by

$$\int\int_S f\,dS=\lim_{\sigma\to 0}\left(\int\int_{S-\sigma} f\,dS\right) \tag{1}$$

b) If the limit in Eq. (1) is finite, the surface integral is said to be *convergent*, otherwise it is *divergent*. If the integral is convergent but the value of the limit depends on the shape of the cavity σ, it is said to be *semiconvergent*.[3]

c) Convergence of the surface integral holds[4] if, within a circle of finite radius whose center is P, the function f is such that

$$|f|<Mr^{-\mu} \tag{2}$$

where $\mu<2$, M is a positive constant, and r is the distance from P to the point at which f is estimated. For a semiconvergent integral, the inequality applies with $\mu=2$.

d) Let Ω be a function with continuous first and second derivatives in S. Let Ω be a function with continuous first and second derivatives in S. Let ψ also be a function with continuous first and second derivatives in S, except possibly at a point P, where it is infinite.

Green's theorem,[3,5] applied to the region bounded by C and the cavity σ which surrounds P, is

$$\int\int_S \left(\psi\nabla^2\Omega-\Omega\nabla^2\psi\right)dS=-\int_C\left(\psi\frac{\partial\Omega}{\partial n}-\Omega\frac{\partial\psi}{\partial n}\right)dc$$
$$-\lim_{\sigma\to 0}\left[\int_\sigma\left(\psi\frac{\partial\Omega}{\partial n}-\Omega\frac{\partial\psi}{\partial n}\right)d\sigma\right] \tag{3}$$

where n is the inward normal at the boundary.

e) Let $\tau=\tau(\xi,\zeta)$ be finite and have uniformly continuous derivatives in the region S. Let $\theta=\theta(\xi-x,\zeta-y)$ be infinite at the point P whose coordinates are (x,y). The following can be proved[3,4]

$$\frac{\partial}{\partial x}\left[\int\int_S \tau(\xi,\zeta)\theta(\xi-x,\zeta-y)\,d\xi d\zeta\right]$$
$$=\int\int_S \tau(\xi,\zeta)\frac{\partial}{\partial x}[\theta(\xi-x,\zeta-y)]\,d\xi d\zeta$$
$$-\tau_p\left\{\lim_{\sigma\to 0}\left[\int_\sigma \theta(\xi-x,\zeta-y)\,n_1 d\sigma\right]\right\} \tag{4}$$

where $n_1=-\cos(n,x)$ is the direction cosine, and τ_p is the value of τ at P.

III. Integral Equation

Consider steady, inviscid, compressible transonic flow past a thin, two-dimensional airfoil of unit length and thickness ratio $\bar{\delta}$, at a small angle-of-attack $\bar{\alpha}$. For subsequent convenience of notation, $\bar{x}$ and $\bar{y}$ are taken as the physical coordinates. It is well known[6] that the governing partial differential equation is

$$[1-M_\infty^2-\Omega_\infty^2(\gamma+1)\bar{\phi}_{\bar{x}}]\bar{\phi}_{\bar{x}\bar{x}}+\bar{\phi}_{\bar{y}\bar{y}}=0 \tag{5}$$

where $\bar{\phi}$ is the perturbation velocity potential, M_∞ is the freestream Mach number, and γ is the ratio of specific heats.

Let $\bar{Y}_+(\bar{x})$ define the upper airfoil profile, and $\bar{Y}_-(\bar{x})$ define the lower profile. Equation (5) is then solved subject to the Kutta condition, the following boundary condition

$$\bar{\phi}_{\bar{y}}(\bar{x},\pm 0)=\frac{d\bar{Y}_\pm}{d\bar{x}} \tag{6}$$

and the condition that $\bar{\phi}_{\bar{x}}$ and $\bar{\phi}_{\bar{y}}$ vanish appropriately at infinity.

In accordance with thin airfoil theory, the airfoil is regarded as a slit on the $\bar{x}$-axis (Fig. 1). Let $\langle\,\rangle$ denote the jump in a quantity across the shock Σ, then[7]

$$\left\langle(1-M_\infty^2)\bar{\phi}_{\bar{x}}-\frac{M_\infty^2(\gamma+1)}{2}(\bar{\phi}_{\bar{x}})^2\right\rangle(d\bar{y})_\Sigma-\langle\bar{\phi}_{\bar{y}}\rangle(d\bar{x})_\Sigma=0 \tag{7a}$$

$$\langle\bar{\phi}_{\bar{y}}\rangle(d\bar{y})_\Sigma+\langle\bar{\phi}_{\bar{x}}\rangle(d\bar{x})_\Sigma=0 \tag{7b}$$

where the subscript Σ denotes an element on the shock surface.

Apply the transformation

$$\beta^2=1-M_\infty^2,\ x=\bar{x},\ y=\beta\bar{y},\ \phi(x,y)=\frac{(\gamma+1)M_\infty^2}{\beta^2}\bar{\phi}(\bar{x},\bar{y}) \tag{8}$$

Equation (5) becomes

$$\phi_{xx}+\phi_{yy}=\phi_x\phi_{xx} \tag{9a}$$

Equation (6) becomes

$$\phi_y(x,\pm 0)=\frac{dY_\pm}{dx} \tag{9b}$$

where $Y_\pm(x)=[(\gamma+1)/\bar{\delta}K^{3/2}]\bar{Y}_\pm(\bar{x})$ defines the modified airfoil profile, and $K=(1-M_\infty^2)/M_\infty{}^{4/3}\bar{\delta}^{2/3}$ is the transonic similarity parameter.

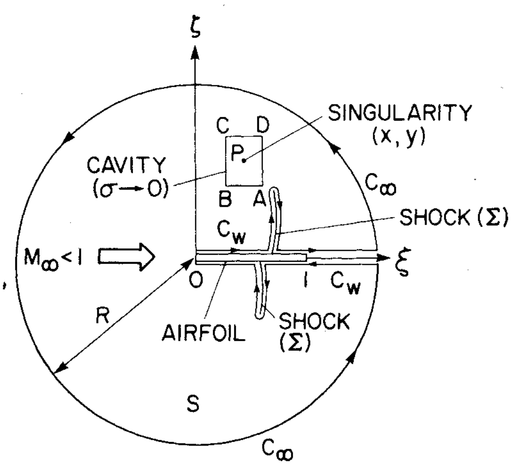

Fig. 1 Region of integration for flow with shocks. For rectangular cavity σ: $AB=2\epsilon$; $DA=2\lambda\epsilon$; $\epsilon\to 0$; aspect ratio $=DA/AB=\lambda$.

Equation (7a) becomes

$$\langle \phi_x - \tfrac{1}{2}(\phi_x)^2 \rangle - \langle \phi_y \rangle \theta_s = 0 \tag{9c}$$

where $\theta_s = (dx/dy)_\Sigma$.

For $M_\infty < 1$, apply Green's theorem to the flowfield (Fig. 1). In Eq. (3), identify Ω with ϕ and ψ with the elementary solution of $\nabla^2\psi = 0$, that is,

$$\psi(\xi - x, \zeta - y) = \ln[(\xi - x)^2 + (\zeta - y)^2]^{1/2} \tag{10}$$

Enclose the singular point of ψ in a rectangular cavity of aspect ratio λ (Fig. 1). Since the airfoil and shocks represent surfaces of discontinuity for some flow quantities, they are excluded from the region of integration as indicated.

Equations (9) can then be converted[4,8,9] to the integro-differential equation

$$\phi(x,y) = \phi_B(x,y) + \frac{1}{2\pi}\int_\Sigma \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)d\Sigma + \frac{1}{2\pi}\int\int_S \psi\phi_\xi\phi_{\xi\xi}d\xi d\zeta \tag{11}$$

where

$$\phi_B(x,y) = \frac{1}{2\pi}\int_0^1 \Big\{\Delta\phi_\zeta(\xi)[\psi(\xi - x, \zeta - y)]_{\zeta=0} - \Delta\phi(\xi)[\psi_\zeta(\xi - x, \zeta - y)]_{\zeta=0}\Big\}d\xi$$

and where

$$\Delta\phi_\zeta(\xi) = \left(\frac{dY_+}{d\xi} - \frac{dY_-}{d\xi}\right) \tag{12b}$$

$$\Delta\phi(\xi) = \phi(\xi, +0) - \phi(\xi, -0) \tag{12c}$$

For a symmetrical airfoil, $Y_-(\xi) = -Y_+(\xi)$. For a nonlifting airfoil, $\Delta\phi(\xi) = 0$.

Integration by parts, of the surface integral, reduces Eq. (11) to[4,8,9]

$$\phi(x,y) = \phi_B(x,y) - \frac{1}{4\pi}\int\int_S \psi_\xi(\xi - x, \zeta - y)[\phi_\xi(\xi,\zeta)]^2 d\xi d\zeta \tag{13}$$

The integral equation is obtained by differentiating both sides of Eq. (13) with respect to x. This involves differentiation, with respect to a parameter, of a singular surface integral.

In Eq. (4), identify τ with ϕ_ξ^2 and θ with ψ_ξ, then

$$\frac{\partial}{\partial x}\Big\{\int\int_S [\phi_\xi(\xi,\zeta)]^2\psi_\xi(\xi - x, \zeta - y)d\xi d\zeta\Big\}$$

$$\int\int_S [\phi_\xi(\xi,\zeta)]^2\psi_{\xi x}(\xi - x, \zeta - y)d\xi d\zeta - [\phi_x(x,y)]^2\Big\{\lim_{\sigma\to 0}\Big[\int_\sigma \psi_\xi(\xi - x, \zeta - y)n_l d\sigma\Big]\Big\} \tag{14a}$$

The line integral around the rectangular cavity can be evaluated to give[4]

$$\lim_{\sigma\to 0}\Big[\int_\sigma \psi_\xi(\xi - x, \zeta - y)n_l d\sigma\Big] = 4\arctan(\lambda) \tag{14b}$$

Differentiate Eq. (13) with respect to x and insert Eqs. (14) to find the integral equation

$$u(x,y) = u_B(x,y) + \nu u^2(x,y) + \int\int_S \kappa(\xi - x, \zeta - y)u^2(\xi,\zeta)dS \tag{15}$$

where

$$\nu = \frac{1}{\pi}\arctan(\lambda)$$

$$u = \frac{\partial\phi}{\partial x}$$

$$u_B = \frac{\partial\phi_B}{\partial x}$$

$$\kappa(\xi - x, \zeta - y) = -\frac{(\xi - x)^2 - (\xi - y)^2}{4\pi[(\xi - x)^2 + (\zeta - y)^2]^2}$$

The integrand of the surface integral in Eq. (15) satisfies Eq. (2) with $\mu = 2$, hence convergence of the integral implies semiconvergence.[4] This means that the value of each of the last two terms in Eq. (15) depends on the shape of the cavity σ, and, hence, on the aspect ratio λ. Their sum, however, is independent of λ, since it represents the derivative of a convergent singular integral. The integral equation derived by Spreiter and Alksne[1] can be deduced from Eq. (15) by choosing $\lambda = \infty$ so that $\nu = \tfrac{1}{2}$.

Enclosing the singularity in a circle is equivalent to enclosing it in a square,[10] hence the integral equation used by Nixon and Hancock[2] can be deduced by choosing $\lambda = 1$ so that $\nu = \tfrac{1}{4}$.

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